MINIMAX ABSORPTION IN A GAME OF ENCOUNTER

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We define a program minimax absorption of a target in a game of encounter of a conflict-controlled motion with a given set and we construct sufficient conditions for the successful solution of this problem.

1. Statement of the problem. We consider an intrinsically linear controlled system described by the vector differential equation

$$dx / dt = A(t)x + f(t, u, v), \qquad u \in P, \quad v \in Q \tag{1.1}$$

Here x is the n-dimensional phase vector of the system, u and v are the r-dimensional control vectors used by the first and second conflicting players, respectively, P and Q are bounded closed sets, the matrix-valued function $A(t)$ and the vector-valued function $f(t, u, v)$ are assumed to be continuous. Let the symbol $\{x\}_m$ denote the vector composed from the first m-coordinates of vector x. Under the conditions of the problem we are given a bounded, closed and convex set M in the space of vectors $\{x\}_m$. An initial position $\{t_0, x_0\}$ is fixed. The first player's aim is to ensure the contact of the point $\{x[t]\}_m$ with set M, while the second player tries to prevent this contact.

Below we shall be interested in the problem only from the standpoint of the first player who needs to ensure contact under all possible actions of the oponent. Let us make this problem, facing the first player, more precise. A function $U(t, x)$ which associates a closed set $U(t, x) \subset P$ with each possible position $\{t, x\}$ $(t \geqslant t_0)$ is called the first player's (pure) strategy $U(t, x)$. The symbol $F(t, x; U)$ denotes the convex hull of the set of all vectors $f(t, u, v)$ for u from $U(t, x)$ and v from Q. A strategy U is said to be admissible if the sets $F(t, x; U)$ are upper-semicontinuous relative to inclusion with respect to the variation of the position $\{t, x\}$. Any absolutely continuous function $x[t]$ satisfying the initial condition $x[t_0] = x_0$ and, for almost all $t \in \in [t_0, \vartheta]$ satisfying the contingencies

$$dx[t] / dt \in A(t)x[t] + F(t, x[t]; U) \tag{1.2}$$

is called the motion $x[t] = x[t; t_0, x_0, U]$ from the position $\{t_0, x_0\}$, generated on the interval $[t_0, \vartheta]$ by the strategy U. The existence of motions $x[t]$ is ensured by the well-known existence theorems for the solutions of the equation in contingencies (1.2) [1]. We say that a strategy U ensures, from the position $\{t_0, x_0\}$, a contact of the motion $x[t]$ of (1.1) with set M at the instant ϑ (by the instant ϑ), if any motion $x[t] = x[t; t_0, x_0, U]$ satisfies the conditions $\{x[\vartheta]\}_m \in M$ ($\{x[t]\}_m \in M$ at least once for $t \in [t_0, \vartheta]$). The first player's problem is to seek ϑ and U such that the strategy U would ensure contact at the instant ϑ (by the instant ϑ).

From the suggested formalization of the problem it follows that methods of forming the control v from a very wide class are admissible. In particular, the choice of the value $v[t]$ can be based on information on the value $u[t]$ realized at this same instant t. To the contrary, the choice of the control $u[t]$ is here made only on the basis of information on the position $\{t, x[t]\}$ realized.

A similar problem for a linear system described by the equation

$$dx / dt = A(t) x + B(t) u - C(t) v$$

was examined in [2] wherein also the appropriate bibliography was cited. The notion, used therein, of a program absorption of the target set M, applied in the linear case for solving the original position game problem of encounter, is here not wholly suitable for the system (1.1) which is nonlinear in u and v if we stay within the framework of pure strategies $U(t, x)$ defined above in correspondence with [2]. If we pass on to the mixed strategies $\{\mu(du) / t, x\}$, then the notion of a program mixed absorption of set M, suitable for the encounter problem in the case of system (1.1), is obtained by an almost automatic transformation of the notion of program absorption in [2]. Here only the program control-functions $u(t)$ and $v(t)$ are replaced by the mixed program control-measures $\mu(du) / t$ and $\nu(dv) / t$ [3]. However, in the framework of pure strategies $U(t, x)$ the transition from the linear system to system (1.1) requires an even more essential transformation of the notion of a program absorption of target M if we have it in mind to use this notion for solving the original position encounter problem from the point of view of the first player's interest. To the contrary, if we have in mind the solution of the position evasion problem, of interest to the second player, the notion of program target absorption, used in [2], which now may be named maximin to distinguish it, remains suitable for this solution also in the case of system (1.1).

The purpose of the present paper is to define the notion of program minimax absorption of set M by process (1.1) and, having delineated the regular case of this absorption, to substantiate the minimax extremal aiming rule which forms the minimax extremal strategy which in this regular case ensures the program minimax absorption of set M from a given initial position $\{t_0, x_0\}$ by an instant ϑ_0. Since the arguments follow much the same plan as in [2], we omit many of the complications and examine in greater detail only those fundamental properties of the program minimax absorption of M being considered now, which distinguish it from the program maximin absorption of M considered in [2].

2. Minimax target absorption. Suppose that a certain number $\vartheta > t_0$ has been chosen. By $V(t, u)$ we denote a function which associates a set $V(t, u) \in Q$ with every pair $\{t, u\}$, where $t \in [t_0, \vartheta]$ and $u \in P$. By $F(t; V)$ we denote the closed convex hull of the set of vectors $f(t, u, v)$ for all v from $V(t, u)$ and u from P. We admit only those functions $V(t, u)$ for which the sets $V(t, u)$ are closed and the sets $F(t; V)$ are upper-semicontinuous relative to inclusion with respect to a change in the variable t on the interval $[t_0, \vartheta)$. We define a motion $x(t) = x(t; t_*, x_*, V)$ as any absolutely continuous function $x(t)$ satisfying the initial condition $x(t_*) = x_*$ and, for almost all t from $[t_*, \vartheta]$, satisfying the contingencies

$$dx(t) / dt \in A(t) x(t) + F(t; V) \tag{2.1}$$

By $G(t_*, x_*, \vartheta; V)$ we denote the attainability region in the space of $\{x\}_m$ ([2], p. 399) from the position $\{t_*, x_*\}$ by the instant $\vartheta \geqslant t_*$ for the motions $x(t) = x(t; t_*, x_*, V)$ of (2.1). It is known that region G is a bounded, convex and closed set.

We say that a program minimax absorption of set M by the process (1.1) at the instant $\vartheta \geqslant t_*$ occurs from the position $\{t_*, x_*\}$ if the regior G intersects M for every choice of an admissible function $V(t, u)$. In other words, a program minimax absorption of set M by the process (1.1) at the instant $\vartheta \geqslant t_*$ occurs from the position $\{t_*, x_*\}$ if and only if for every choice of an admissible function $V(t, u)$ at least one motion $x(t)=x(t; t_*, x_*, V)$ of (2.1) hits onto M at the instant ϑ, i.e., the inclusion $\{x(\vartheta)\}_m \in M$ is realized. By M_ε we denote closed Euclidean ε- neighborhoods of set M. Let $\varepsilon_0(t_*, x_*, \vartheta)$ be the lower bound of those values of $\varepsilon \geqslant 0$ for which there occurs the program minimax absorption of set M_ε by the process (1.1) at the instant ϑ from the position $\{t_*, x_*\}$. By analogy with [2] (p. 108) we name the quantity $\varepsilon_0(t, x, \vartheta)$ the minimax hypothetical mismatch. Let $G(t_*, x_*, \vartheta; V)_0$ be that attainability region $G(t_*, x_*, \vartheta; V)$ whose distance from M equals $\varepsilon_0(t_*, x_*, \vartheta)$. (For the present we assume a priori the existence of such nonempty regions G_0 under the condition $\varepsilon_0 > 0$; we verify their existence later on). We call a case regular if for every choice of position $\{t_*, x_*\}$ at which $\varepsilon_0(t_*, x_*, \vartheta) > 0$ all regions G_0 intersect M_{ε_0} along one single hyperplane. For a position $\{t_*, x_*\}$ the number $\vartheta_0 \geqslant t_*$ is called the (first) instant of program minimax absorption of set M by process (1.1) if this number ϑ_0 is the smallest of the numbers $\vartheta \geqslant t_*$ for which there occurs a program minimax absorption of set M by the process (1.1) at the instant ϑ from the position $\{t_*, x_*\}$.

3. Minimax hypothetical mismatch. We set up an expression for computing the quantity ε_0. Suppose that a certain admissible function $V(t, u)$ has been chosen. The region $G(t_*, x_*, \vartheta; V)$ intersects the set M_ε if and only if the closed "$(-M_\varepsilon)$-neighborhood" $G(t_*, x_*, \vartheta; V)_{(-M_\varepsilon)}$ of the region $G(t_*, x_*, \vartheta; V)$ contains the point $\{x\}_m = 0$. (The region $G_{(-M_\varepsilon)}$ is made up of all vectors $q = g - h + k$, where $g \in G$, $h \in M$ and $\|k\| \leqslant \varepsilon$, where $\|k\|$ denotes the Euclidean norm of the vector.) But a convex bounded closed set $G_{(-M_\varepsilon)}$ is the intersection of its support halfspaces H_l (see Fig. 1)

$$l'\{x\}_m \geqslant \min_q l'q, \quad q \in G(t_*, x_*, \vartheta; V)_{(-M_\varepsilon)} \tag{3.1}$$

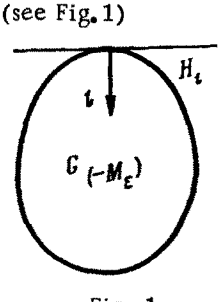

Fig. 1.

Here l is the m-dimensional unit vector, and the prime denotes the transpose.

By the definition of the set $G(t_*, x_*, \vartheta; V)_{(-M_\varepsilon)}$ and of the attainability region $G(t_*, x_*, \vartheta; V)$ and in correspondence with (2.1), for $q \in G_{(-M_\varepsilon)}$ we have

$$q = \{x(\vartheta)\}_m - h + k =$$
$$= \left\{X(\vartheta, t_*)x_* + \int_{t_*}^{\vartheta} X(\vartheta, t)w(t)\,dt\right\}_m - h + k \tag{3.2}$$

Here $X(t, \tau)$ is the fundamental matrix of solutions of the equation

$$dx/dt = A(t)x, \quad w(t) \in F(t; V), \quad h \in M, \quad \|k\| \leqslant \varepsilon$$

From (3.1) and (3.2) it follows that the point $\{x\}_m = 0$ lies in the region $G(t_*, x_*, \vartheta; V)_{(-M_\varepsilon)}$ if and only if the condition

$$l'\{X(\vartheta, t_*)x_*\}_m + \min_{w(t)} \int_{t_*}^{\vartheta} l'\{X(\vartheta, t)w(t)\}_m\,dt +$$
$$+ \min_h l'(-h) - \varepsilon \leqslant 0, \quad w(t) \in F(t; V), \quad h \in M \tag{3.3}$$

is fulfilled for any unit vector l. From (3.3) it follows that the quantity $\varepsilon_0 = \varepsilon_0(t_*, x_*, \vartheta)$ is determined by the relation

$$\varepsilon_0 = \sup_{\|l\|=1}\Big(\Big[\sup_V \min_{w(t)} \int_{t_*}^{\vartheta} l' \{X(\vartheta, t) w(t)\}_m dt\Big] + + l'\{X(\vartheta, t_*) x_*\}_m - \max_h l'h \tag{3.4}$$

if the right-hand side of this equality is nonnegative; otherwise, $\varepsilon_0(t_*, x_*, \vartheta) = 0$.
Let us verify that the upper bound over all admissible functions $V(t, u)$ ocurring within the brackets in the right hand side of (3.4), is achieved on some admissible function $V_l(t, u)$. To do this it suffices to choose as the sets $V_l(t, u)$ those sets which are made up of all vectors v_u from Q satisfying the condition

$$l'\{X(\vartheta, t) f(t, u, v_u)\}_m = \max_{v\in Q}(l'\{X(\vartheta, t) f(t, u, v)\}_m) \tag{3.5}$$

Indeed, the function $V_l(t, u)$ is admissible because we can verify that the sets $V_l(t, u)$ defined by condition (3.5) are closed and upper-semicontinuous relative to inclusion with respect to changes in t and u. Furthermore, for every value of u and for any v, from (3.5) follows the inequality

$$l'\{X(\vartheta, t) f(t, u, v_u)\}_m \geqslant l'\{X(\vartheta, t) f(t, u, v)\}_m$$

From this inequality follows

$$\min_{u\in P}(l'\{X(\vartheta, t) f(t, u, v_u)\}_m) \geqslant \min_{u\in P}(l'\{X(\vartheta, t) f(t, u, v)\}_m) \tag{3.6}$$

or, by definition of $F(t; V)$,

$$\min_{w\in F(t;\, V_l)}(l'\{X(\vartheta, t) w\}_m) \geqslant \min_{w\in F(t;\, V)}(l'\{X(\vartheta, t) w\}_m) \tag{3.7}$$

for any admissible function V.

The left-hand sides of (3.7) and (3.6) equal the quantity

$$\min_{u\in P}\max_{v\in Q}(l'\{X(\vartheta, t) f(t, u, v)\}_m) = \varkappa(t, \vartheta, l)$$

It is a continuous function of variable t. From this and from the properties of the sets $F(t; V_l)$ follows the existence of a measurable function $w_l(t) \in F(t; V_l)$, which for almost all t satisfies the condition

$$l'\{X(\vartheta, t) w_l(t)\}_m = \min_{w\in F(t;\, V_l)}(l'\{X(\vartheta, t) w\}_m) \tag{3.8}$$

Further, from (3.7) and the properties of sets $F(t; V)$ it follows that whatever be the admissible function $V(t, u)$ we can find a measurable function $w(t) \in F(t; V)$ for almost all t, for which

$$\min_{w\in F(t;\, V_l)}(l'\{X(\vartheta, t) w\}_m) \geqslant l'\{X(\vartheta, t) w(t)\}_m$$

By comparing the inequalities obtained, we arrive at the needed relation

$$\int_{t_*}^{\vartheta} l'\{X(\vartheta, t) w_l(t)\}_m dt = \min_{w(t)\in F(t;\, V_l)}\Big(\int_{t_*}^{\vartheta} l'\{X(\vartheta, t) w(t)\}_m dt\Big) \geqslant$$

$$\geqslant \sup_V \min_{w(t)\in F(t;\, V)}\Big(\int_{t_*}^{\vartheta} l'\{X(\vartheta, t) w(t)\}_m dt\Big)$$

This relation proves that the upper bound with respect to V in (3.4) indeed is achieved by an admissible function $V_l\,(t,\ u)$. Moreover, it also follows from the arguments presented that the quantity $\varepsilon_0\,(t_*,\ x_*,\ \vartheta)$ in (3.4) is determined by the equality

$$\varepsilon_0 = \max_{\|l\|=1}\Big(\int\limits_{t_*}^{\vartheta} \min_{u\in P}\max_{v\in Q} l'\,\{X\,(\vartheta, t)\,f\,(t, u, v)\}_m\,dt + \\ + l'\,\{X\,(\vartheta, t_*)\,x_*\}_m - \max_{h\in M} l'h\Big) \qquad (3.9)$$

if the right-hand side of this equality is nonnegative; otherwise, $\varepsilon_0\,(t_*, x_*, \vartheta) = 0$. The arguments presented show also that when $\varepsilon_0\,(t_*,\ x_*,\ \vartheta) > 0$ there indeed do exist regions $G\,(t_*,\ x_*,\ \vartheta;\ V)_0$, whose distance from M equals the quantity ε_0. These attainability regions are generated by the admissible functions $V = V_l\,(t,\ u)$ corresponding to those values of l which maximize the right-hand side of (3.9). Further, as in [2], from (3.9) we conclude that a case is regular if and only if for $\varepsilon_0 > 0$ the maximum in the right-hand side of (3.9) is reached only on the unit vector $l^\circ\,(t_*,\ x_*,\ \vartheta)$. For this, in its own turn, it is necessary and sufficient that the quantity being maximized in (3.9), taken with the opposite sign, be a convex function of the variable l.

Finally, be repeating with necessary but minor changes the discussion from [2] (p.149) we can verify that in the regular case the quantity $\varepsilon_0\,(t, x, \vartheta)$ is a differentiable function of the variables t and x in the region $\varepsilon_0\,(t,\ x,\ \vartheta) > 0$ for a fixed value of ϑ, and its continuous partial derivatives $\partial\varepsilon_0\,/\,\partial t$, $\partial\varepsilon_0\,/\,\partial x_i$ at the point $\{t,\ x\}$ are determined by the relations

$$\frac{\partial\varepsilon_0}{\partial t} = -\min_{u\in P}\max_{v\in Q}\,[s'(t)\,f\,(t, u, v)] - s'\,(t)A\,(t)x$$

$$\partial\varepsilon_0\,/\,\partial x_i = s_i\,(t) \qquad (i = 1, 2, \ldots, n) \qquad (3.10)$$

Here $s\,(\tau)$ is a vector-valued function satisfying the conditions

$$ds\,(\tau)\,/\,d\tau = -A'\,(\tau)\,s\,(\tau) \qquad (t \leqslant \tau \leqslant \vartheta) \qquad (3.11)$$

$$s'\,(\vartheta) = (\overset{\circ}{l}_1, \ldots, \overset{\circ}{l}_m, 0, \ldots, 0)$$

moreover, $l^\circ\,(t,\ x,\ \vartheta)$ is the vector which solves problem (3.9) (for $t = t_*,\ \ x = x_*$).

4. Minimax extremal aiming. We consider the regular case. Suppose that a position $\{t_*,\ x_*\}$ is realized at some instant $t = t_*$ and, moreover, that $\varepsilon_0\,(t_*,\ x_*,\ \vartheta) > 0$ for some instant ϑ fixed beforehand. From the material in Sect. 3 it follows that there exists a certain attainability region $G\,(t_*,\ x_*,\ \vartheta;\ V_{l^\circ})$ which is tangent to the ε_0-neighborhood M_{ε_0} of set M, and every other attainability region $G\,(t_*,\ x_*,\ \vartheta;\ V)$ for the motion $x\,(t;\ t_*,\ x_*,\ V)$ of (2.1) also necessarily intersects M_{ε_0}. Let $\{x\}_m = g_0$ be one of the points in the space of $\{x\}_m$ at which the sets M_{ε_0} and $G\,(t_*, x_*, \vartheta;\ V_{l^\circ})$ intersect. By the regularity condition all such points lie on one hyperplane $l^{\circ\prime}(t_*,\ x_*,\ \vartheta)\ \{x\}_m = \alpha$. Since, furthermore, the point g_0 lies on the boundary of the region $G\,(t_*,\ x_*,\ \vartheta;\ V_{l^\circ})$, the motion $x\,(t;\ \ t_*,\ x_*,\ V_{l^\circ})$ of (2.1), which has been led to this point at the instant $t = \vartheta$, is optimal and the control $w^\circ\,(t) \in \in F\,(t;\ V_{l^\circ})\ (t_* \leqslant t \leqslant \vartheta)$ generating it should satisfy the appropriate condition of the maximum principle [4]. In the case being considered it is convenient to write this condition as a minimum condition

$$s'\,(t)\,w^\circ\,(t) = \min_{w\in F\,(t;\ V_{l^\circ})}\,[s'\,(t)\,w] \qquad (4.1)$$

where s (t) is the vector-valued function defined by conditions (3.11). The maximum condition has been replaced by a minimum condition for the reason that the vector l° determining the boundary condition in (3.11), has the sense of not the outward but the inward normal to the attainability region G (Fig. 1).

Since $s'(t) = (l_1^\circ, \ldots, l_m^\circ, 0, \ldots, 0)\, X(\vartheta, t)$, the function $w_0(t)$ from (3.8) with $l = l^\circ(t_*, x_*, \vartheta)$ just satisfies condition (4.1).

Now, by carrying over to the case being considered the definition in [2] (pp.115-121) of extremal aiming, we say that every control w_e from the set of all values $w_e(t_*) \in \in F(t_*; V_{l^\circ})$, which satisfy condition (4.1) with $t = t_*$, effects a minimax extremal aiming of the motion of (2.1) from the position $\{t_*, x_*\}$ towards one of the points g_v by the instant ϑ. In correspondence with this, the minimax extremal strategy U_e is given by the set $U_e(t, x)$, which at each position $\{t_*, x_*\}$ for $\varepsilon_0(t_*, x_*, \vartheta) > 0$ are made up of all values of u_e satisfying the minimax condition

$$\min_{u\in P} \max_{v\in Q} [s'(t_*) f(t_*, u, v)] = \max_{v\in Q} [s'(t_*) f(t_*, u_e, v)] \qquad (4.2)$$

while for $\varepsilon_0(t_*, x_*, \vartheta) = 0$ we set $U_e(t, x) = P$.

It can be verified that an extremal strategy is admissible. Further, from conditions (4.2) and from expressions (3.10) it follows that the equality

$$\max_{x[t]} \left(\frac{d\varepsilon_0(t, x[t], \vartheta)}{dt} \right)_{U_e} = \min_U \max_{x[t]} \left(\frac{d\varepsilon_0(t, x[t], \vartheta)}{dt} \right)_U = 0 \qquad (4.3)$$

is fulfilled in the region $\varepsilon_0 > 0$. Here the symbol $(d\varepsilon_0(t, x[t], \vartheta) / dt)_U$ denotes the total time derivative of the function $\varepsilon_0(t, x[t], \vartheta)$ along the motion $x[t]$ of system (1.1) under a chosen strategy U, i.e., the total derivative with respect to time t of the function $\varepsilon_0(t, x[t], \vartheta)$ along any solution $x[t]$ of the equations in contingencies (1.2). Then, by repeating with appropriate changes the discussions in [2] (p.153), we can convince ourselves of the validity of the following assertion.

Theorem. Suppose that for the initial position $\{t_0, x_0\}$ there exists the instant ϑ_0 of program minimax absorption of set M and that for this value of $\vartheta = \vartheta_0$ the case is regular. Then for the fixed value of $\vartheta = \vartheta_0$ the extremal strategy $U_e(t, x)$ ensures the contact of the motion $x[t] = x[t; t_0\, x_0, U_e]$ with the set M by the instant ϑ_0, so that for every motion $x[t] = x[t; t_0, x_0, U_e]$ we have $\{x[\vartheta_0]\}_m \in M$.

Note. If in the course of things a position $\{t_*, x_*\}$ is realized for which $\vartheta_0(t_*, x_*) < < \vartheta_0(t_0, x_0)$, then, beginning with the instant t_*, we can pass on to an extremal strategy U_e corresponding now to a new value of the absorption instant $\vartheta_0(t_*, x_*)$ (under the assumption that the case remains regular).

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